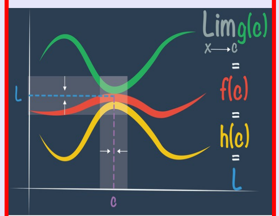
Calculus I

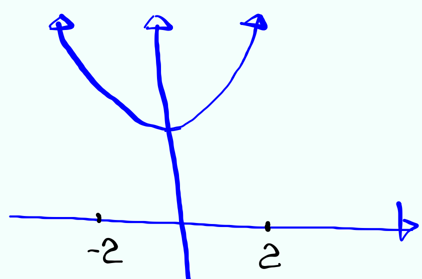
Lecture 24



Feb 19-8:47 AM

Class QZ 19

Find $\boxed{f_{ave}}$ of the function $\boxed{f(x) = 3x^2+4}$ over the interval $\boxed{[-2,2]}$.



$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

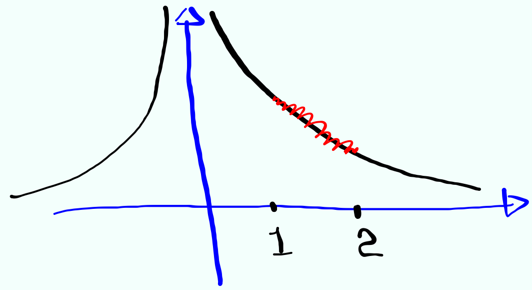
$$= \frac{1}{2-(-2)}\int_{-2}^{2}(3x^2+4)\,dx = \frac{1}{4}\cdot 2\int_0^2 (3x^2+4)\,dx$$

$$= \frac{1}{2}\left[x^3+4x\right]\Big|_0^2 = \frac{1}{2}\left[2^3+4(2) - 0\right] = \frac{1}{2}\cdot 16 = \boxed{8}$$

Jul 28-7:22 AM

Find f_{ave} of $f(x) = \frac{1}{x^2}$ on $[1,2]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



$$= \frac{1}{2-1}\int_1^2 \frac{1}{x^2}\,dx$$

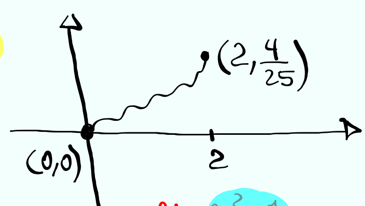
$$= \frac{1}{1}\int_1^2 x^{-2}\,dx = \frac{x^{-2+1}}{-2+1}\Big|_1^2 = \frac{-1}{x}\Big|_1^2 = -\left(\frac{1}{2} - \frac{1}{1}\right)$$

$$= -\left(\frac{-1}{2}\right) = \boxed{\frac{1}{2}}$$

Jul 28-8:17 AM

Find f_{ave} of $f(x) = \frac{2x}{(x^2+1)^2}$ on $[0,2]$.

$$f_{ave} = \frac{1}{2-0}\int_0^2 \frac{2x}{(x^2+1)^2}\,dx$$



$$= \frac{1}{2}\int_1^5 \frac{1}{u^2}\,du$$

$u = x^2+1$

$du = 2x\,dx$

$x=0 \to u=1$

$x=2 \to u=5$

$$= \frac{1}{2}\int_1^5 u^{-2}\,du$$

$$= \frac{1}{2}\left[\frac{u^{-2+1}}{-2+1}\right]\Big|_1^5$$

$$= \frac{-1}{2}\,\frac{1}{u}\Big|_1^5 = \frac{-1}{2}\left[\frac{1}{5} - \frac{1}{1}\right] = \frac{-1}{2}\left[\frac{1}{5} - 1\right]$$

$$= \frac{-1}{2}\left[\frac{-4}{5}\right] = \boxed{\frac{2}{5}}$$

Jul 28-8:22 AM

find f_{ave} of $f(x) = (x-3)^2$ on $[2,5]$.

$$f_{ave} = \frac{1}{5-2}\int_2^5 (x-3)^2\,dx$$

$u = x-3$

$du = dx$

$x=2 \rightarrow u=-1$

$x=5 \rightarrow u=2$

$$= \frac{1}{3}\int_{-1}^{2} u^2\,du$$

$$= \frac{1}{3}\,\frac{u^3}{3}\Bigg|_{-1}^{2} = \frac{1}{9}\left[2^3-(-1)^3\right]$$

$$= \frac{1}{9}\left[8-(-1)\right] = \boxed{1}$$

Jul 28-8:30 AM

find f_{ave} for $f(x) = (x-2)^4$ on $[0,4]$.

$$f_{ave} = \frac{1}{4-0}\int_0^4 (x-2)^4\,dx$$

$u = x-2$

$du = dx$

$x=0 \rightarrow u=-2$

$x=4 \rightarrow u=2$

$$= \frac{1}{4}\int_{-2}^{2} u^4\,du$$

$$= \frac{1}{4}\cdot 2\int_0^2 u^4\,du$$

$$= \frac{1}{2}\cdot\frac{u^5}{5}\Bigg|_0^2 = \boxed{\frac{32}{10}}$$

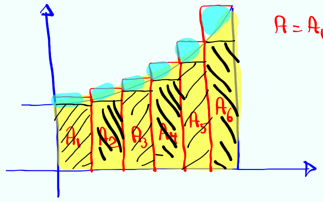
If $f(x)$ is an even function: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

If $f(x)$ is an odd function: $\int_{-a}^{a} f(x)\,dx = 0$

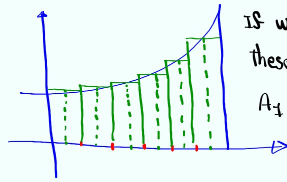
$$\int_a^a f(x)\,dx = 0$$

Jul 28-8:39 AM

Find the area shaded below



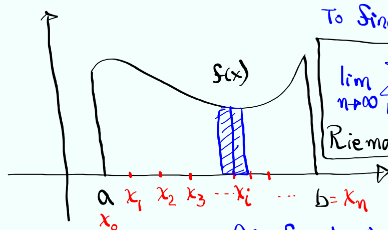
$A = A_1 + A_2 + A_3 + \cdots + A_6$



If we get infinite # of these rectangles

$A_1 + A_2 + A_3 + \cdots + A_n$ when $n \to \infty$

Find the area below $f(x) \geq 0$, above x-axis, from $x = a$ to $x = b$.



To find total Area

$$\boxed{\lim_{n \to \infty} \sum_{i=1}^{n} A_i}$$

Riemann Sum

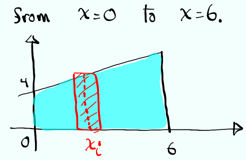
Area of ith rectangle: $A_i = f(x_i) \cdot \Delta x$

Jul 28-8:50 AM

Riemann Sum

$$\boxed{A = \lim_{n \to \infty} \sum_{i=1}^{n} A_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x \qquad \Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x}$$

Find the area below $f(x) = x + 4$, above x-axis from $x = 0$ to $x = 6$.



Trapezoid

$h = 6$, $b = 4$, $B = 10$

$A = \frac{h(B+b)}{2} = \frac{6(10+4)}{2} = \boxed{42}$ ✓

$a = 0$ $b = 6$

$\Delta x = \frac{b-a}{n} = \frac{6-0}{n} = \frac{6}{n}$

$x_i = a + i\Delta x$

$= 0 + i \cdot \frac{6}{n} = \frac{6i}{n}$

Riemann Sum

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \left[\frac{6i}{n} + 4\right] \frac{6}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{36i}{n^2} + \frac{24}{n}\right) = \lim_{n \to \infty} \left[\sum_{i=1}^{n} \frac{36i}{n^2} + \sum_{i=1}^{n} \frac{24}{n}\right]$$

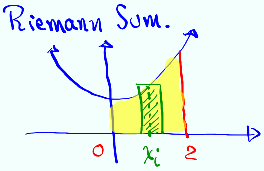
$$= \lim_{n \to \infty} \left[\frac{36}{n^2} \sum_{i=1}^{n} i + \frac{24}{\cancel{n}} \cdot \cancel{n}\right] = \lim_{n \to \infty} \left[\frac{\overset{18}{\cancel{36}}}{n^2} \cdot \frac{n(n+1)}{\cancel{2}} + 24\right]$$

$$= \lim_{n \to \infty} \left[\frac{18n(n+1)}{n^2} + 24\right] = \lim_{n \to \infty} \left(\frac{18n^2 + \cdots}{n^2} + 24\right)$$

$$= 18 + 24 = \boxed{42}$$

Jul 28-9:02 AM

Find the area below $f(x)=x^2+4$, above x-axis from $x=0$ to $x=2$ using Riemann Sum.



Integration

$$A=\int_0^2 (x^2+4)\,dx$$
$$=\left(\frac{x^3}{3}+4x\right)\Big|_0^2$$
$$=\frac{8}{3}+4(2)-0$$
$$=\boxed{\frac{32}{3}}$$

$$A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

$a=0$ $\quad b=2$ $\quad \Delta x=\frac{b-a}{n}=\frac{2}{n}$ $\quad x_i=a+i\Delta x=\frac{2i}{n}$

$$f(x_i)=\left(\frac{2i}{n}\right)^2+4=\frac{4i^2}{n^2}+4$$

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}+4\right)\cdot\frac{2}{n}=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{8i^2}{n^3}+\frac{8}{n}\right)$$

$$=\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{8i^2}{n^3}+\sum_{i=1}^{n}\frac{8}{n}\right]=\lim_{n\to\infty}\left[\frac{8}{n^3}\sum_{i=1}^{n}i^2+\frac{8}{n}\cdot n\right]$$

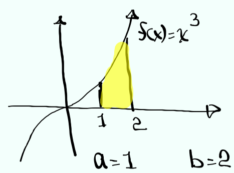
$$=\lim_{n\to\infty}\left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+8\right]=\lim_{n\to\infty}\left[\frac{16n^3+\cdots}{6n^3}+8\right]$$

$$=\frac{16}{6}+8=\frac{8}{3}+8=\boxed{\frac{32}{3}}$$

Jul 28-9:17 AM

If $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$ exists, it is equal to $\int_a^b f(x)\,dx$

where
$\Delta x=\frac{b-a}{n}$
$x_i=a+i\Delta x$



$$A=\int_1^2 x^3\,dx=\frac{x^4}{4}\Big|_1^2$$
$$=\frac{1}{4}\left[2^4-1^4\right]=\frac{15}{4}\ \checkmark$$

$a=1$ $\quad b=2$ $\quad \Delta x=\frac{b-a}{n}=\frac{1}{n}$

$$x_i=a+i\Delta x=1+i\cdot\frac{1}{n}=1+\frac{i}{n}$$

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\cdot\frac{1}{n}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{3i}{n}+\frac{3i^2}{n^2}+\frac{i^3}{n^3}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n\cdot 1+\frac{3}{n}\cdot\frac{n(n+1)}{2}+\frac{3}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^3}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to\infty}\left[1+\frac{3n^2+\cdots}{2n^2}+\frac{6n^3+\cdots}{6n^3}+\frac{n^4+\cdots}{4n^4}\right]$$

$$=1+\frac{3}{2}+\frac{6}{6}+\frac{1}{4}=1+1\tfrac{1}{2}+1+\frac{1}{4}$$
$$=3\tfrac{3}{4}=\boxed{\frac{15}{4}}$$

Jul 28-9:30 AM

Evaluate $\int_0^1 (3x-1)^{50}\,dx$

Use Subs. Method

$u = 3x-1$

$du = 3dx$

$\frac{du}{3} = dx$

$x=0 \rightarrow u=-1$

$x=1 \rightarrow u=2$

$= \int_{-1}^{2} u^{50}\,\frac{du}{3}$

$= \frac{1}{3}\cdot \frac{u^{51}}{51}\Big|_{-1}^{2}$

$= \frac{1}{153}\, u^{51}\Big|_{-1}^{2} = \frac{1}{153}\left[2^{51} - (-1)^{51}\right]$

$= \boxed{\frac{2^{51}+1}{153}}$

Jul 28-10:45 AM

Find $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$= \int \frac{\sin u}{\cancel{u}}\,2\cancel{u}\,du$

$= 2\int \sin u\,du = -2\cos u + C$

$= \boxed{-2\cos\sqrt{x} + C}$

Evaluate $\int_0^{\sqrt{\pi/2}} x\cos x^2\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$x=0 \rightarrow u=0$

$x=\sqrt{\frac{\pi}{2}} \rightarrow u=\frac{\pi}{2}$

$= \int_0^{\pi/2} \cos u\,\frac{du}{2}$

$= \frac{1}{2}\int_0^{\pi/2} \cos u\,du$

$= \frac{1}{2}\cdot \sin u\Big|_0^{\pi/2} = \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= \boxed{\frac{1}{2}}$

Jul 28-10:50 AM

find $\int x\sqrt{x+1}\,dx$

$u-1=x$

$u=x+1$

$du=dx$

$$=\int (u-1)\sqrt{u}\,du=\int (u\sqrt{u}-\sqrt{u})\,du$$

$$=\int (u^{3/2}-u^{1/2})\,du$$

$$=\frac{u^{5/2}}{5/2}-\frac{u^{3/2}}{3/2}+C$$

$$=\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+C$$

Jul 28-10:59 AM

Evaluate $\int_0^1 \frac{1}{(1+\sqrt{x})^4}\,dx$

$u=1+\sqrt{x}$

$u-1=\sqrt{x}$

$(u-1)^2=x$

$u^2-2u+1=x$

$(2u-2)\,du=dx$

$$=\int_1^2 \frac{2u-2}{u^4}\,dx$$

$$=\int_1^2 \left(\frac{2u}{u^4}-\frac{2}{u^4}\right)du$$

$$=2\int_1^2 (u^{-3}-u^{-4})\,du=2\left[\frac{u^{-2}}{-2}-\frac{u^{-3}}{-3}\right]\Bigg|_1^2=\boxed{\frac{1}{6}}$$

Make sure to verify.

Jul 28-11:03 AM